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Central Elements of Atomic Effect Algebras

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Various conditions ensuring that an atomic effect algebra is a Boolean algebra are presented.

KEY WORDS: effect algebra; Boolean algebra; central element; atomic; atomistic.

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Central elements and the center (the set of all central elements) play an important role in quantum structures—they represent the "classical part" of a given model. Considering the axiomatics of quantum structures, it is important to know the impact of various conditions on the size of the center. In particular, there are a lot of results of the type that a quantum structure with some properties has to be a Boolean algebra—see, e.g., Navara and Pták (1989), Müller *et al.* (1992), Pulmannová and Majerník (1992), Müller (1993), Pulmannová (1993), Pták and Pulmannová (1994), Tkadlec (1994), Dvurečenskij and Länger (1995), Tkadlec (1995, 1997), Navara (1997). All the results mentioned above were generalized by Tkadlec (2004) by a characterization of a central element of an effect algebra. In this paper, we present conclusions of the results presented at the latter paper for atomic effect algebras.

1. BASIC NOTIONS AND PROPERTIES

Let us summarize some basic notions and properties of effect algebras. For proofs and details see, e.g., Foulis and Bennett (1994), Greechie *et al.* (1995).

Definition 1.1. An *effect algebra* is an algebraic structure $(E, 0, 1, \oplus)$ such that E is a set, 0 and 1 are different elements of E and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$, the following conditions hold (the equalities mean also "if one side exists then the other side exists"):

(1) $a \oplus b = b \oplus a$ (*commutativity*),

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(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (*associativity*),
(3) for every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$ (*orthosupplement*),
(4) $a = 0$ whenever $a \oplus 1$ is defined (*zero-unit law*).

For simplicity, we use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \leq b$ iff there is a $c \in E$ such that $b = a \oplus c$; such an element c is unique (if it exists) and is denoted by $b \ominus a$. 0 (1, resp.) is the least (the greatest, resp.) element of E with respect to this partial ordering. An *orthogonality* relation on E is defined by $a \perp b$ iff $a \oplus b$ exists (i.e., iff $a \leq b'$). It can be shown that $a \oplus 0 = a$ for every $a \in E$ and that a *cancellation law* is valid: for every $a, b, c \in E$ with $a \oplus b \leq a \oplus c$ we have $b \leq c$.

Obviously, if $a \perp b$ and $a \vee b$ exists in an effect algebra, then $a \vee b \leq a \oplus b$. The reverse inequality need not be true.

Definition 1.2. An element a of an effect algebra E is *principal*, if $b \oplus c \leq a$ for every $b, c \in E$ such that $b, c \leq a$ and $b \perp c$.

If a is a principal element, then $a \wedge a' = 0$ and the interval $[0, a]$ is an effect algebra with the greatest element a and the partial operation given by restriction of $\oplus$ to $[0, a]$—the orthosupplement operation is given by $b \mapsto (b \oplus a')'$.

Our interest will be concentrated on central elements of atomic effect algebras.

Definition 1.3. An element a of an effect algebra E is *central*, if

(1) a and a' are principal,
(2) for every $b \in E$ there are $b_1, b_2 \in E$ such that $b_1 \leq a$, $b_2 \leq a'$ and $b = b_1 \oplus b_2$.

The *center* $C(E)$ of E is the set of all central elements of E.

The center of an effect algebra E is a sub-effect algebra of E and forms a Boolean algebra. The decomposition property of central elements (condition (2) of Definition 1.3.) can be formulated by the following way: $b = (b \wedge a) \oplus (b \wedge a')$. Central elements correspond to direct product decompositions of effect algebras.

Definition 1.4. An *atom* of an effect algebra E is a minimal element of $E \setminus \{0\}$. A *coatom* of an effect algebra is the orthosupplement of an atom. An effect algebra is *atomic*, if every nonzero element dominates an atom (i.e., there is an atom less than or equal to it). An effect algebra is *atomistic*, if every nonzero element is a supremum of a set of atoms (i.e., of the set of all atoms it dominates).

Obviously, every atomistic effect algebra is atomic. On the contrary, not every atomic effect algebra is atomistic—see, e.g., Greechie (1969) or Example 2.4. (Let us remark that atomic orthomodular lattice is atomistic.)

2. RESULTS

Let us start with the characterization of central elements in effect algebras proved in Tkadlec (2004).

Theorem 2.1. *Let E be an effect algebra. Then $a \in E$ is central iff the following conditions hold:*

(1) *a and a' are principal,*
(2) *$b = 0$ whenever $b \in E$ with $b \wedge a = b \wedge a' = 0$,*
(3) *$[0, a] \cap [0, b]$, $[0, a'] \cap [0, b]$ have maximal elements for every $b \in E$.*

The condition (2) of Theorem 2.1 is a week form of distributivity—it can be reformulated by the following way: $b \wedge (a \vee a') = (b \wedge a) \vee (b \wedge a')$ whenever $b \wedge a = b \wedge a' = 0$. Let us discuss this condition for atomic effect algebras.

Proposition 2.2. *Let E be an effect algebra. Let us consider the following conditions:*

(W) *For all $a, b \in E$, if $a \wedge b = a \wedge b' = 0$, then $a = 0$.*
(W$'$) *For all $a, b \in E$, if a is an atom, then either $a \leq b$ or $a \leq b'$.*
(W$''$) *For all $a, b \in E$, if a, b are atoms and $a \neq b$, then $a \perp b$.*

The condition (W) *implies the condition* (W$'$) *which implies the condition* (W$''$).

If the effect algebra E is atomic, then the condition (W$'$) *implies the condition* (W).

If the effect algebra E is atomistic, then the condition (W$''$) *implies the condition* (W$'$).

Proof: (W)⇒(W$'$): Let $a, b \in E$ and let a be an atom. If $a \not\leq b$, then $a \wedge b = 0$. Since $a \neq 0$, we obtain, according to condition (W), that it is not true that $a \wedge b' = 0$. Since a is an atom, it means that $a \leq b'$.

(W$'$)⇒(W$''$): Let $a, b \in E$ be distinct atoms. Then $a \not\leq b$ and, according to condition (W$'$), $a \leq b'$, i.e., $a \perp b$.

(W$'$)⇒(W) if E is atomic: We prove that if condition (W) is not fulfilled then condition (W$'$) is also not fulfilled. Let us suppose that there are elements $a, b \in E$, $a \neq 0$, such that $a \wedge b = a \wedge b' = 0$. Since the effect algebra E is atomic, there is an atom $c \in E$ such that $c \leq a$. Hence, $c \wedge b = c \wedge b' = 0$ and therefore $c \not\leq b$ and $c \not\leq b'$.

(W$''$)$\Rightarrow$(W$'$) if E is atomistic: Let $a, b \in E$ and let a be an atom. Let us suppose that $a \not\leq b$. If $b = 0$ then $b' = 1 \geq a$. Let us consider the case $b \neq 0$. Since the effect algebra E is atomistic, there is a set $A_b \subset E$ of atoms such that $b = \vee A_b$. Since $a \notin A_b$, we obtain, according to condition (W$''$), that $a \perp c$ for every element $c \in A_b$, i.e., $c \leq a'$ for every element $c \in A_b$. Hence, $b = \vee A_b \leq a'$, i.e., $a \leq b'$. □

As a consequence, in atomistic effect algebras all these conditions are equivalent, in atomic effect algebras the conditions (W) and (W$'$) are equivalent. Let us show that condition (W$'$) does not imply condition (W) for nonatomic effect algebras (Example 2.3) and that condition (W$''$) does not imply condition (W$'$) for nonatomistic effect algebras (Example 2.4).

Example 2.3. Let X be an infinite set and let B be the Boolean algebra of all subsets of X factorized over finite subsets of X (i.e., we "identify" subsets of X with finite symmetric difference). Let us consider the so-called *horizontal sum* E of two copies of B, i.e., the union of disjoint copies of B (also with the $\oplus$ operation) and identify the least and the greatest elements of both copies. More formally, we consider the Cartesian product $B \times \{0, 1\}$ and for a equal to the least or the greatest element of B we put $(a, 0) = (a, 1)$. The effect algebra E (it is even an ortomodular poset) does not contain any atom, hence the condition (W$'$) is fulfilled. On the other side, for every $a, b \in B$ that are neither minimal nor maximal elements of B, we obtain that $(a, 0) \wedge (b, 1) = (a, 0) \wedge (b', 1)$ is the minimal element of E. Hence, the condition (W) is not fulfilled.

Example 2.4. Let X_1, X_2, X_3, X_4 be pairwise disjoint sets, X be their union, and suppose X_3, X_4 be infinite. Let us put $E^* = \{X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, \emptyset, X\}$ and let E consist of all subsets S of X for which there is an element $S^* \in E^*$ such that the symmetric difference of S and S^* is a finite subset of $X_3 \cup X_4$. Let $a \oplus b = a \cup b$ if a and b are disjoint. Then E is an effect algebra (even an orthomodular poset), the partial order is the set-theoretic inclusion, and the atoms are $X_1 \cup X_2$ and one-element subsets of $X_3 \cup X_4$. Hence, the condition (W$''$) is fulfilled. Since for the atom $a = X_1 \cup X_2$ and the element $b = X_1 \cup X_4$ we have $a \not\leq b$ and $a \not\leq b' = X_2 \cup X_4$, the condition (W$'$) is not fulfilled.

The following statement is a consequence of Theorem 2.1 and Proposition 2.2.

Corollary 2.5. *An effect algebra E is a Boolean algebra if the following conditions hold:*

(1) *every element of E is principal,*
(2) *at least one of the following conditions holds:*
  (2′) *E is atomic, and $a \le b$ or $a \le b'$ for every atom $a \in E$ and every $b \in E$,*
  (2″) *E is atomistic, and $a \perp b$ for every pair of distinct atoms in E,*
(3) *$[0, a] \cap [0, b]$ has a maximal elements for every $a, b \in E$.*

We derived the previous result using the meaning of the condition (2) of Theorem 2.1 in atomic effect algebras. However, we may try another attempt—to study when an atom is central and when the centrality of atoms implies the centrality of all elements of an effect algebra. Let us start with the properties of atoms.

Lemma 2.6. *Let E be an effect algebra and let $a \in E$ be an atom.*

(1) *The atom a is principal iff $a \not\perp a$ (a is not isotropic).*

(2) *If the coatom a' is principal, then for every element $b \in E$ either $a \not\le b$ or $a \not\le b'$ (i.e., a is principal).*

Proof:

1) If an element $a \in E$ is both isotropic and principal, then $a \oplus a \le a = a \oplus 0$ and using the cancellation law we obtain $a \le 0$. Hence, a principal atom is not isotropic. Let an atom a be not isotropic and let $b, c \in E$, $b, c \le a$ and $b \perp c$. Since a is an atom, we have $b, c \in \{0, a\}$. Since a is not isotropic, we have $\{b, c\} \ne \{a\}$. Hence, either $b = c = 0$ and then $b \oplus c = 0 \le a$ or $\{b, c\} = \{0, a\}$ and then $b \oplus c = a \le a$.

2) Let us suppose that there is an element $b \in E$ such that $a \le b$ and $a \le b'$. Hence $b, b' \le a'$ and $b \oplus b' = 1 > a'$ and therefore a' is not principal. (The principality of a follows from putting $b = a$, obtaining thus $a \not\le a'$ and considering part (1) of this lemma.) □

Lemma 2.7. *Let E be an effect algebra, $a, b \in E$, and a be an atom. Then the following conditions are equivalent:*

(1) *There are $b_1, b_2 \in E$ such that $b_1 \le a$, $b_2 \le a'$ and $b = b_1 \oplus b_2$.*
(2) *$a \le b$ or $a \le b'$.*

Proof: Let us suppose that the condition (1) holds and that $a \not\le b$. Hence, $b_1 = 0$, $b = 0 \oplus b_2 = b_2 \le a'$ and therefore $a \le b'$.

If $a \leq b$ then $b = a \oplus (b \ominus a)$, $(b \ominus a) \perp a$ and therefore $(b \ominus a) \leq a'$. If $a \leq b'$ then $b = 0 \oplus b$, $b \leq a'$. □

The following statement is a consequence of Theorem 2.1, Lemma 2.6 and Lemma 2.7.

Proposition 2.8. *An atom a of an effect algebra E is central iff the following conditions hold:*

(1) *a' is principal,*
(2) *$a \leq b$ or $a \leq b'$ for every element $b \in E$.*

(*The "or" at the second condition might be considered exclusive.*)

The following proposition (slightly reformulated here) was published by Foulis and Bennett (1994, Theorem 4.11).

Proposition 2.9. *If E is an atomic effect algebra such that every subset of E has a maximal element, then every nonzero element of E is a finite sum of atoms.*

Proof: Let us suppose that an element $b \in E \setminus \{0\}$ is not a finite sum of atoms and seek a contradiction. Since E is atomic, there is an atom $a_1 \in E$ such that $a_1 \leq b$. Since b is not a finite sum of atoms, we have $b \ominus a_1 \neq 0$ and therefore there is an atom $a_2 \in E$ such that $a_2 \leq a \ominus a_1$. Since b is not a finite sum of atoms, we have $b \ominus (a_1 \oplus a_2) \neq 0$ and therefore there is an atom $a_3 \in E$ such that $a_3 \leq a \ominus (a_1 \oplus a_2)$. Continuing in this procedure, we obtain a sequence of (not necessarily distinct) atoms $a_1, a_2, a_3, \ldots \in E$ such that $a_1 < a_1 \oplus a_2 < a_1 \oplus a_2 \oplus a_3 < \cdots < b$, hence the set $\{a_1, a_1 \oplus a_2, a_1 \oplus a_2 \oplus a_3, \ldots\}$ does not have a maximal element—a contradiction. □

The following statement is a consequence of Propositions 2.8 and 2.9.

Corollary 2.10. *An effect algebra E is a Boolean algebra if the following conditions hold:*

(1) *every coatom is principal,*
(2) *E is atomic, and $a \leq b$ or $a \leq b'$ for every atom $a \in E$ and every $b \in E$,*
(3) *every subset of E has a maximal element.*

Using two different ways we came to results with similar conditions. Combining these results we obtain the following statement with a bit complicated structure. It seems to be an open question whether there might be introduced some reasonable notions such that the formulation will be more simple.

Theorem 2.11. *An effect algebra E is a Boolean algebra if the following conditions hold:*

(1) *at least one of the following conditions hold:*
   (1′) *every element is principal, and* $[0, a] \cap [0, b]$ *has a maximal element for every* $a, b \in E$,
   (1″) *E is atomic, every coatom is principal, and every subset of E has a maximal element,*
(2) *at least one of the following conditions hold:*
   (2′) $b = 0$ *whenever* $a, b \in E$ *with* $b \wedge a = b \wedge a' = 0$,
   (2″) *E is atomic, and $a \leq b$ or $a \leq b'$ for every atom $a \in E$ and every $b \in E$,*
   (2‴) *E is atomistic, and $a \perp b$ for every pair of distinct atoms in E.*

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